Weil divisors (Har II 6)

HW: Read all examples in II 6.

Let X be a scheme. Recall that X is **regular** at $x \in X$ if

$\#$ generators of $m_x = \dim \mathcal{O}_x$.

$\shortparallel$

$\left(\dim_{k(x)} m_x / m_x^2\right)$

Def: X is regular in codimension one if every local ring $\mathcal{O}_x$ of dimension one is regular (hence a DVR).

Geometrically, this means that the singular locus has codimension > 1.

Ex: If $X = \operatorname{Spec} A$, (Noetherian) $\dim \mathcal{O}_{X,p} = 1 \iff \operatorname{codim} P = 1$. So X is regular iff A_p is a DVR for every codim 1 prime.
(This will automatically be satisfied if A is normal. — see A-M)

To define Weil divisors, for the remainder of the section, we'll assume:

(*) X is a Noetherian integral separated scheme which is regular in codim one.
↳ e.g. Noetherian normal schemes

Def: A prime divisor on X is a closed integral subscheme Y of codimension one. A Weil divisor is an element of $\operatorname{Div} X$, which is defined

$$\operatorname{Div} X := \left\{ \sum_{i=1}^{n} a_i P_i \;\middle|\; P_i \subseteq X \text{ is a prime divisor, } a_i \in \mathbb{Z},\ n \geq 0 \right\}$$

i.e. the free abelian group generated by the prime divisors.

If $D = \sum a_i P_i$ is a Weil divisor and $a_i \geq 0$ for all i, then D is <u>effective</u>.

Since X is integral, recall from HW#3 that the <u>function field</u> of X is $K(X) = \mathcal{O}_{X,\xi}$, the stalk at the generic point ξ. You showed that $K(X)$ is a field and if $U = \operatorname{Spec} A \subseteq X$ is open, then $K(X) =$ field of fractions of A.

Let $Y \subseteq X$ be a prime divisor. Then by assumption $\mathcal{O}_{X,Y}$ (stalk at generic point of Y) is a DVR.

Note that $\mathcal{O}_{X,\eta} = \varinjlim_{U \cap Y \neq \emptyset} \mathcal{O}_X(U) \hookrightarrow K(X)$.
$K(X)$ is the field of fractions of $\mathcal{O}_{X,\eta}$:

Let $U = \operatorname{Spec} A \subseteq X$ be an open affine meeting Y (i.e. containing the generic point of Y). Then Y corresponds to a height one (codim one) prime $P \in \operatorname{Spec} A$.

Then $\mathcal{O}_{X,\eta} = A_P$ (do you see why?), which has field of fractions $K(X)$.

The corresponding valuation $v_Y : K(X)^* \to \mathbb{Z}$ is called the valuation of Y.

We can write elements of $K(X)$ as f/g where $f, g \in A$, and $v_Y(f/g) = v(f) - v(g)$. Recall that $v(f) =$ highest power of max'l ideal of A_P containing f.

Ex: In $\mathbb{P}^2_{\mathbb{C}}$ let $V = V(x^3 - y^2 z)$. $D_+(z)$ meets V, and it corresponds to $P = (x^3 - y^2 z) = \left(\left(\frac{x}{z}\right)^3 - \left(\frac{y}{z}\right)^2 \right) = (X^3 - Y^2)$, where $\frac{x}{z} = X$, $\frac{y}{z} = Y$.

$K(\mathbb{P}^2) = K(\mathbb{A}^2) = \mathbb{C}\left(\frac{x}{z}, \frac{y}{z}\right)$.

$\mathcal{O}_{\mathbb{P}^2, V} = \mathbb{C}[X, Y]_P$ is the local ring w/ max'l ideal $(X^3 - Y^2)$.

$v_V\left(\frac{z^3}{x^3}\right) = 0$ since this is a unit.

$v_V\left(\frac{z^3}{x^3 - y^2 z}\right) = -1$, since it is the inverse of a uniformizing parameter.

More generally, if $f \in K(X)^*$ and $Y \subseteq X$ a prime divisor,

$v_Y(f) > 0 \iff f$ has a zero along Y of that order

$v_Y(f) < 0 \iff f$ has a pole along Y of that order.

(so e.g. X^2Y has a zero along $V(X)$ of order 2.)

Lemma: Let X be a scheme and $f \in K(X)^*$ nonzero. Then $v_Y(f) = 0$ for all but finitely many prime divisors $Y \subseteq X$.

Pf: Let $U = \operatorname{Spec} A$ be an open affine on which f is regular. Set $Z = X \setminus U$, a closed subset. X is Noetherian, so Z can contain at most finitely many prime divisors (Do you see why?). All the others meet U.

Thus, it suffices to show that there are only finitely many prime divisors Y of U for which $v_Y(f) \neq 0$.

Since f is regular on U, $f \in A$, so $v_Y(f) \geq 0$, and $v_Y(f) > 0$ iff $f \in P$, where P is the codim 1 prime defining Y. These correspond to minimal primes of $A/(f)$, of which there are only finitely many. (Equivalently, $V(f)$ can only contain finitely many closed irred. subsets of U of codim 1.) $\square$

Elements of the function field define Weil divisors on X as follows:

Def: Let $f \in K(X)^*$. Define the divisor of f, denoted (f), by

$$(f) := \sum_{\substack{Y \subseteq X \\ \text{prime} \\ \text{divisor}}} v_Y(f) \cdot Y$$

(By the lemma, this is a finite sum.) Any divisor of this form is called a <u>principal divisor</u>.

Note that if $f, g \in K^*$, then $(f/g) = (f) - (g)$.

In particular, $K^* \longrightarrow \operatorname{Div} X$, defined $f \longmapsto (f)$ is a group homomorphism, with image the subgroup of principal divisors. The cokernel of this map is called the <u>divisor class group of X</u>, denoted $\operatorname{Cl} X$.

<u>Def:</u> D, $D' \in \operatorname{Div} X$ are <u>linearly equivalent</u>, written $D \sim D'$, if $D - D'$ is a principal divisor. That is, if D and D' have the same image in $\operatorname{Cl} X$.

In general, $\operatorname{Cl} X$ is hard to calculate, but we can calculate it in some examples:

<u>Ex:</u> $X = \operatorname{Spec} A$, A a UFD. We know from CA that A is a UFD $\Leftrightarrow$ all codim 1 primes are principal. Thus, if Y is a prime divisor, then the corresponding codim 1 prime P is principal. Thus, all prime divisors are principal, so all divisors are. Thus, $\operatorname{Cl} X = 0$.

In fact, the converse holds when X is normal!

Def: An integral scheme X is <u>normal</u> if $\mathcal{O}_x$ is integrally closed for all $x \in X$.

Note that $\operatorname{spec} A$ is normal $\Leftrightarrow$ A_p is int. closed $\Leftrightarrow$ A is int. closed

check!
Hint: $A = \cap A_p$

Prop: Let A be a Noetherian integral domain. Then A is a UFD $\Leftrightarrow$ $X = \operatorname{Spec} A$ is normal and $\operatorname{Cl} X = 0$.

Pf: ($\Rightarrow$) If A is a UFD, then A is integrally closed, and we already showed that $\operatorname{Cl} X = 0$.

($\Leftarrow$) Let $P \subseteq A$ be a codim 1 prime. We need to show P is principal. Let $Y \subseteq X$ be the corresponding prime divisor.

Since $\operatorname{Cl} X = 0$, there's some $f \in K$ s.t. $(f) = Y$. Thus, $v_Y(f) = 1$, so f generates the maximal ideal $PA_p \subseteq A_p$.

Let $P' \subseteq A$ be another prime ideal of codim one and Y' the corresponding divisor. Then $v_{Y'}(f) = 0$, so $f \in A_{p'}$.

Thus, $f \in \bigcap_{\operatorname{codim} P = 1} A_P \overset{\text{comm. alg result}}{=} A$. In particular, $f \in A \cap PA_p = P$.

So we just need that $(f) = P$. Let $g \in P$. Then $v_Y(g) \geq 1$ and $v_{Y'}(g) \geq 0$ for any prime divisor Y' (since $g \in A \subseteq A_{p'}$)

Thus, $v_{Y'}(g/f) \geq 0$ for all Y' $\Rightarrow$ $g/f \in$ all $A_{p'}$ $\Rightarrow$ $g/f \in A$ so f divides $g \Rightarrow g \in (f) \Rightarrow (f) = P$. $\square$

Example: Divisors on $\mathbb{P}^n_k$:

Let $X = \mathbb{P}^n_k$, $D = \sum n_i Y_i \in \operatorname{Div} X$. We define the degree of D to be

$$\deg D = \sum n_i \deg Y_i$$

where $\deg Y_i$ = degree of generator of homog. ideal of Y_i.

If $f \in K^*$, then $f = \frac{G}{H}$, G, H homogeneous of the same degree d. Thus, G and H both define a union of hypersurfaces whose degrees sum to 0. Thus,

$$(f) = \underbrace{\sum n_i Y_i}_{\text{hypersurfaces defined by } G} - \underbrace{\sum m_j Z_j}_{\text{hypsfs defined by } H}$$

where $\deg(f) = \sum n_i \deg Y_i - \sum m_j \deg Z_j = 0$.

Moreover, any divisor of degree 0 is principal. To see this, let D is a divisor and $\deg D = d$. We write

$$D = D_1 - D_2$$

where D_1, D_2 are effective of degrees d_1, d_2 so that $d = d_1 - d_2$. Let G_i be the homog. poly. describing D_i.

(We can find such G_i since an irred. hypersurface in $\mathbb{P}^n$ corr. to a codim one homog. prime in $k[x_0, \dots, x_n]$, which is principal. Let G_i be the product of the corresponding generators.)

By abuse of notation, write $(G_i) = D_i$, so that $D = (G_1) - (G_2)$.

Define $f = \frac{G_1}{G_2 x_0^d}$. Then $(f) = \overbrace{(G_1) - (G_2)}^{D} - dH$, where H is the hyperplane $x_0 = 0$.

$\Rightarrow D \sim dH$. In particular, if $\deg D = 0$, then $D \sim 0$, so D is principal. This yields the following:

Prop: $\mathrm{Cl}\, \mathbb{P}^n_k = 0$.

Pf: Let $\deg : \mathrm{Div}\, X \to \mathbb{Z}$ be the degree function, which is clearly a homomorphism. We showed that the kernel is exactly the set of principal divisors. Thus it induces an isom. $\deg : \mathrm{Cl}\, X \xrightarrow{\cong} \mathbb{Z}$. $\square$

Ex: Let X be a scheme and $Z \subsetneq X$ closed. Set $U = X - Z$.

If $Y \subseteq X$ is a prime divisor, then $Y \cap U$ is either empty or a prime divisor on U (Note it can't have greater codim: $\mathcal{O}_{X,Y} = \mathcal{O}_{U,Y}$ when the generic point of Y is in U.) Thus, we have a homomorphism

$$\begin{aligned} \mathrm{Cl}\, X &\longrightarrow \mathrm{Cl}\, U \\ \sum n_i Y_i &\longmapsto \sum n_i (Y_i \cap U) \end{aligned}$$

where we ignore empty $Y_i \cap U$. Note that this is well-defined since if $f \in K(X)^*$, then $f \in K(U)^*$, so $(f) \mapsto (f)$.

Every prime divisor of U is the intersection of U w/ some prime divisor in X, so the map is surjective.

If $\operatorname{codim} Z \geq 2$, then Z can't contain any prime divisor of X, so the map is an isomorphism.

If $\operatorname{codim} Z = 1$ and Z is integral, then the kernel of the map is generated by Z. That is

$$\begin{aligned} \mathbb{Z} &\longrightarrow \mathrm{Cl}\, X \longrightarrow \mathrm{Cl}\, U \longrightarrow 0 \quad \text{is exact.} \\ 1 &\longmapsto Z \end{aligned}$$

Ex: If $Z \subseteq \mathbb{P}^2_k$ is a curve of degree d, and $U = \mathbb{P}^2_k - Z$, then we have $\mathrm{Cl}\, X \cong \mathbb{Z}$ and the image of Z is d, so $\mathrm{Cl}\, U \cong \operatorname{coker}\left(\begin{aligned} \mathbb{Z} &\to \mathbb{Z} \\ 1 &\mapsto d \end{aligned} \right) \cong \mathbb{Z}/d\mathbb{Z}$.

Claim $\mathrm{Cl}\, X \cong \mathrm{Cl}(X \times \mathbb{A}^1)$.

$$X \times_{\mathbb{Z}} \mathbb{A}^1_{\mathbb{Z}}$$

Pf: See Har 6.6.

We'll use this to calculate the class group in the following example.

Ex: Let Q be the nonsingular quadric surface $xy = zw$ in $\mathbb{P}^3_k$. Recall that Q is the image of the Segre embedding

$$\mathbb{P}^1_k \times_k \mathbb{P}^1_k \hookrightarrow \mathbb{P}^3_k$$

(See I Ex 2.15 in Hartshorne. Do this problem if you've never done it before!)

Let p_1, p_2 be the projections of Q onto the two factors. Then for each p_i, we define a map

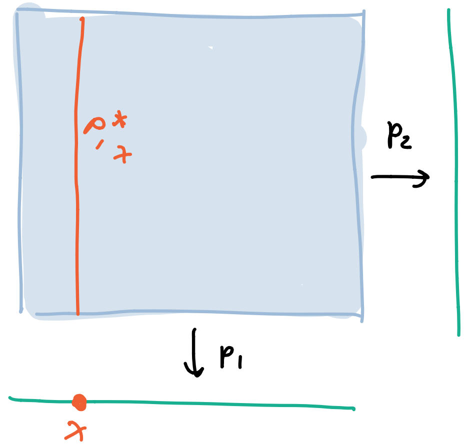
$$\mathrm{Cl}\, \mathbb{P}^1 \longrightarrow \mathrm{Cl}(Q)$$
$$D = \sum n_i Y_i \longmapsto p_i^* D := \sum n_i p_i^{-1}(Y_i)$$

Note that since the generic point of Q maps to the generic point of $\mathbb{P}^1$, p_i induces an inclusion of fields

$$K(\mathbb{P}^1) \hookrightarrow K(Q)$$

So if $f \in K(\mathbb{P}^1)^*$, we have $p_i^*((f)) = (f)$, making the map a homomorphism.

Let $Y = \{x\} \times \mathbb{P}^1$, $x \in \mathbb{P}^1$ some point.



Then $Q - Y = \mathbb{A}^1 \times \mathbb{P}^1$ and we get a composition

$$\mathrm{Cl}\,\mathbb{P}^1 \xrightarrow{p_2^*} \mathrm{Cl}\,Q \longrightarrow \mathrm{Cl}(\mathbb{A}^1 \times \mathbb{P}^1)$$

which is the isomorphism from above. Thus p_2^* (and p_1^*) are injective. We also have the exact sequence

$$\begin{array}{ccccccc} \mathbb{Z} & \longrightarrow & \mathrm{Cl}\,Q & \longrightarrow & \mathrm{Cl}(\mathbb{A}^1 \times \mathbb{P}^1) & \longrightarrow & 0. \\ 1 & \longmapsto & Y & & & & \end{array}$$

But $\mathbb{Z} \cong \mathrm{Cl}\,\mathbb{P}^1$, $1 \mapsto pt$, so this map is just p_1^*, so it's injective. Thus, we have

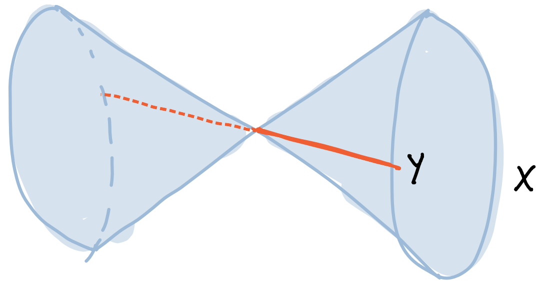
$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \arrow[r, "p_1^*"] & \mathrm{Cl}\,Q \arrow[r] & \mathrm{Cl}(\mathbb{A}^1 \times \mathbb{P}^1) \arrow[r] & 0 \\ & & \mathbb{Z} \arrow[u, "p_2^*"] \arrow[ur, "\cong"'] & & \end{tikzcd}$$

So by the splitting lemma, $\mathrm{Cl}\,Q = \mathrm{Im}\,p_1^* \oplus \mathrm{Im}\,p_2^* \cong \mathbb{Z} \oplus \mathbb{Z}$.

For any divisor D in $\mathrm{Cl}\,Q$, $D \sim (a, b)$, some $a, b \in \mathbb{Z}$. We say D has <u>type</u> (a, b).

Ex: Let $A = k[x,y,z]/(xy - z^2)$, and let X be the singular quadric surface $X = \operatorname{Spec} A$.

Let Y be cut out by the ideal (y,z).
Note that Y has codim 1 since A has dim 2 and $(y,z) \subsetneq (x,y,z)$. Thus, Y is a prime divisor, so we have



$$\mathbb{Z} \to Cl\, X \to Cl(X-Y) \to 0$$
$$1 \mapsto Y$$

Consider the local ring $\mathcal{O}_{X,Y} = A_{(y,z)}$. $\frac{x}{1}$ is a unit, so the max'l ideal is $(y,z) = (z^2, z) = (z)$.

Then the divisor $(y) = 2 \cdot Y$ since $y \in (z)^2$ in $\mathcal{O}_{X,Y}$.

Note that set theoretically, $V(y,z) = V(y)$, since $z^2 \in (y)$, so $(y) = (y, z^2)$. Thus,

$$X - Y = \operatorname{Spec} A_y.$$

$$A_y = k[x,y,y^{-1},z]/(xy - z^2) \Rightarrow x = \frac{z^2}{y}, \text{ so}$$

$A_y \cong k[y, y^{-1}, z]$, which is a UFD, so $Cl(X-Y) = 0$.

Thus, we have $\mathbb{Z} \twoheadrightarrow Cl\, X$, which is generated by Y, and $2Y = 0$. Thus, we have either $Cl\, X = 0$ or $\mathbb{Z}/2\mathbb{Z}$.

However, A is integrally closed (check!) but not a UFD.

Thus, $\operatorname{cl} X \neq 0$, so $\operatorname{cl} X = \mathbb{Z}/2\mathbb{Z}$.